

ON THE COMMUTING VARIETY OF A REDUCTIVE LIE ALGEBRA

JEAN-YVES CHARBONNEL

ABSTRACT. The commuting variety of a reductive Lie algebra \mathfrak{g} is the underlying variety of a well defined subscheme of $\mathfrak{g} \times \mathfrak{g}$. In this note, it is proved that this scheme is normal. In particular, its ideal of definition is a prime ideal.

CONTENTS

1. Introduction	1
2. Characteristic module	3
3. Torsion and projective dimension	6
4. Main results	12
Appendix A. Projective dimension and cohomology	14
References	16

1. INTRODUCTION

In this note, the base field \mathbb{k} is algebraically closed of characteristic 0, \mathfrak{g} is a reductive Lie algebra of finite dimension, ℓ is its rank, and G is its adjoint group.

1.1. Notations. • For V a module over a \mathbb{k} -algebra, its symmetric and exterior algebras are denoted by $S(V)$ and $\wedge(V)$ respectively. If E is a subset of V , the submodule of V generated by E is denoted by $\text{span}(E)$. When V is a vector space over \mathbb{k} , the grassmanian of all d -dimensional subspaces of V is denoted by $\text{Gr}_d(V)$.

• All topological terms refer to the Zariski topology. If Y is a subset of a topological space X , let denote by \overline{Y} the closure of Y in X . For Y an open subset of the algebraic variety X , Y is called *a big open subset* if the codimension of $X \setminus Y$ in X is bigger than 2. For Y a closed subset of an algebraic variety X , its dimension is the biggest dimension of its irreducible components and its codimension in X is the smallest codimension in X of its irreducible components. For X an algebraic variety, $\mathbb{k}[X]$ is the algebra of regular functions on X .

• All the complexes considered in this note are graded complexes over \mathbb{Z} of vector spaces and their differentials are homogeneous of degree -1 and they are denoted by d . As usual, the gradation of the complex is denoted by C_\bullet .

Date: March 8, 2013.

1991 Mathematics Subject Classification. 14A10, 14L17, 22E20, 22E46 .

Key words and phrases. polynomial algebra, complex, commuting variety, Cohen-Macaulay, homology, projective dimension, depth.

- The dimension of the Borel subalgebras of \mathfrak{g} is denoted by $b_{\mathfrak{g}}$. Let set $n := b_{\mathfrak{g}} - \ell$ so that $\dim \mathfrak{g} = 2b_{\mathfrak{g}} - \ell_{\mathfrak{g}} = 2n + \ell$.
- The dual \mathfrak{g}^* of \mathfrak{g} identifies with \mathfrak{g} by a given non degenerate, invariant, symmetric bilinear form $\langle \cdot, \cdot \rangle$ on $\mathfrak{g} \times \mathfrak{g}$ extending the Killing form of $[\mathfrak{g}, \mathfrak{g}]$.
- For $x \in \mathfrak{g}$, let denote by \mathfrak{g}^x the centralizer of x in \mathfrak{g} . The set of regular elements of \mathfrak{g} is

$$\mathfrak{g}_{\text{reg}} := \{x \in \mathfrak{g} \mid \dim \mathfrak{g}^x = \ell\}$$

The subset $\mathfrak{g}_{\text{reg}}$ of \mathfrak{g} is a G -invariant open subset of \mathfrak{g} . According to [V72], $\mathfrak{g} \setminus \mathfrak{g}_{\text{reg}}$ is equidimensional of codimension 3.

• Let denote by $S(\mathfrak{g})^{\mathfrak{g}}$ the algebra of \mathfrak{g} -invariant elements of $S(\mathfrak{g})$. Let p_1, \dots, p_{ℓ} be homogeneous generators of $S(\mathfrak{g})^{\mathfrak{g}}$ of degree d_1, \dots, d_{ℓ} respectively. Let choose the polynomials p_1, \dots, p_{ℓ} so that $d_1 \leq \dots \leq d_{\ell}$. For $i = 1, \dots, \ell$ and $(x, y) \in \mathfrak{g} \times \mathfrak{g}$, let consider a shift of p_i in direction y : $p_i(x + ty)$ with $t \in \mathbb{k}$. Expanding $p_i(x + ty)$ as a polynomial in t , one obtains

$$(1) \quad p_i(x + ty) = \sum_{m=0}^{d_i} p_i^{(m)}(x, y) t^m; \quad \forall (t, x, y) \in \mathbb{k} \times \mathfrak{g} \times \mathfrak{g}$$

where $y \mapsto (m!)p_i^{(m)}(x, y)$ is the derivate at x of p_i at the order m in the direction y . The elements $p_i^{(m)}$ defined by (1) are invariant elements of $S(\mathfrak{g}) \otimes_{\mathbb{k}} S(\mathfrak{g})$ under the diagonal action of G in $\mathfrak{g} \times \mathfrak{g}$. Let remark that $p_i^{(0)}(x, y) = p_i(x)$ while $p_i^{(d_i)}(x, y) = p_i(y)$ for all $(x, y) \in \mathfrak{g} \times \mathfrak{g}$.

Remark 1.1. The family $\mathcal{P}_x := \{p_i^{(m)}(x, \cdot); 1 \leq i \leq \ell, 1 \leq m \leq d_i\}$ for $x \in \mathfrak{g}$, is a Poisson-commutative family of $S(\mathfrak{g})$ by Mishchenko-Fomenko [MF78]. One says that the family \mathcal{P}_x is constructed by the *argument shift method*.

- Let $i \in \{1, \dots, \ell\}$ be. For x in \mathfrak{g} , let denote by $\varepsilon_i(x)$ the element of \mathfrak{g} given by

$$\langle \varepsilon_i(x), y \rangle = \frac{d}{dt} p_i(x + ty) \big|_{t=0}$$

for all y in \mathfrak{g} . Thereby, ε_i is an invariant element of $S(\mathfrak{g}) \otimes_{\mathbb{k}} \mathfrak{g}$ under the canonical action of G . According to [Ko63, Theorem 9], for x in \mathfrak{g} , x is in $\mathfrak{g}_{\text{reg}}$ if and only if $\varepsilon_1(x), \dots, \varepsilon_{\ell}(x)$ are linearly independent. In this case, $\varepsilon_1(x), \dots, \varepsilon_{\ell}(x)$ is a basis of \mathfrak{g}^x .

Let denote by $\varepsilon_i^{(m)}$, for $0 \leq m \leq d_i - 1$, the elements of $S(\mathfrak{g} \times \mathfrak{g}) \otimes_{\mathbb{k}} \mathfrak{g}$ defined by the equality:

$$(2) \quad \varepsilon_i(x + ty) = \sum_{m=0}^{d_i-1} \varepsilon_i^{(m)}(x, y) t^m, \quad \forall (t, x, y) \in \mathbb{k} \times \mathfrak{g} \times \mathfrak{g}$$

and let set:

$$V_{x,y} := \text{span}(\{\varepsilon_i^{(0)}(x, y), \dots, \varepsilon_i^{(d_i-1)}(x, y), i = 1, \dots, \ell\})$$

for (x, y) in $\mathfrak{g} \times \mathfrak{g}$.

1.2. Main result and main idea. Since \mathfrak{g} identifies with its dual, $S(\mathfrak{g})$ is the algebra of polynomial functions on \mathfrak{g} . The commuting variety $\mathcal{C}(\mathfrak{g})$ of \mathfrak{g} is the subvariety of elements (x, y) of $\mathfrak{g} \times \mathfrak{g}$ such that $[x, y] = 0$. Let $I_{\mathfrak{g}}$ be the ideal of $S(\mathfrak{g} \times \mathfrak{g})$ generated by the functions $(x, y) \mapsto \langle v, [x, y] \rangle$ with v in \mathfrak{g} . Then $\mathcal{C}(\mathfrak{g})$ is the underlying subvariety of the subscheme of $\mathfrak{g} \times \mathfrak{g}$ defined by $I_{\mathfrak{g}}$. It is a well known and long standing open question whether or not this scheme is reduced. The main result of this note is the following theorem:

Theorem 1.2. *The subscheme of $\mathfrak{g} \times \mathfrak{g}$ defined by $I_{\mathfrak{g}}$ is normal. Furthermore, $I_{\mathfrak{g}}$ is a prime ideal of $S(\mathfrak{g} \times \mathfrak{g})$.*

According to a result of R. W. Richardson [Ri79], $\mathcal{C}(\mathfrak{g})$ is irreducible. So the last assertion of the theorem is a consequence of the first one. In [Di79], J. Dixmier gave a partial answer to this question. In fact, he proved that $I_{\mathfrak{g}}$ and its radical have the same part in degree 1. The main tool of our proof uses the main argument of the Dixmier's proof: for a finitely generated module M over $S(\mathfrak{g} \times \mathfrak{g})$, $M = 0$ if the codimension of its support is at least $l + 2$ with l the projective dimension of M (see Appendix A).

Let introduce the characteristic submodule of \mathfrak{g} , denoted by $B_{\mathfrak{g}}$. By definition, $B_{\mathfrak{g}}$ is a submodule of $S(\mathfrak{g} \times \mathfrak{g}) \otimes_{\mathbb{K}} \mathfrak{g}$ and an element φ of $S(\mathfrak{g} \times \mathfrak{g}) \otimes_{\mathbb{K}} \mathfrak{g}$ is in $B_{\mathfrak{g}}$ if and only if for all (x, y) in $\mathfrak{g} \times \mathfrak{g}$, $\varphi(x, y)$ is in the sum of subspaces \mathfrak{g}^{ax+by} with (a, b) in $\mathbb{K}^2 \setminus \{0\}$. This is a free module of rank $b_{\mathfrak{g}}$. Moreover, for all φ in $B_{\mathfrak{g}}$ and for all (x, y) in $\mathfrak{g} \times \mathfrak{g}$, $\langle \varphi(x, y), [x, y] \rangle = 0$. The first step of the proof of Theorem 1.2 is the following proposition:

Proposition 1.3. *For i positive integer, the module $\wedge^i(\mathfrak{g}) \wedge \wedge^{b_{\mathfrak{g}}} (B_{\mathfrak{g}})$ has projective dimension at most i .*

The second step of the proof of Theorem 1.2 is the following theorem:

Theorem 1.4. *The ideal $I_{\mathfrak{g}}$ is radical and its projective dimension is $2n - 1$.*

Then Theorem 1.2 follows easily from this theorem and [Po08, Theorem 1].

2. CHARACTERISTIC MODULE

For (x, y) in $\mathfrak{g} \times \mathfrak{g}$, let set:

$$V'_{x,y} = \sum_{(a,b) \in \mathbb{K}^2 \setminus \{0\}} \mathfrak{g}^{ax+by}$$

By definition, the characteristic module $B_{\mathfrak{g}}$ of \mathfrak{g} is the submodule of elements φ of $S(\mathfrak{g} \times \mathfrak{g}) \otimes_{\mathbb{K}} \mathfrak{g}$ such that $\varphi(x, y)$ is in $V'_{x,y}$ for all (x, y) in $\mathfrak{g} \times \mathfrak{g}$. In this section, some properties of $B_{\mathfrak{g}}$ are given.

2.1. Let denote by $\Omega_{\mathfrak{g}}$ the subset of elements (x, y) of $\mathfrak{g} \times \mathfrak{g}$ such that $P_{x,y}$ has dimension 2 and such that $P_{x,y} \setminus \{0\}$ is contained in $\mathfrak{g}_{\text{reg}}$. According to [CMo08, Corollary 10], $\Omega_{\mathfrak{g}}$ is a big open subset of $\mathfrak{g} \times \mathfrak{g}$.

Proposition 2.1. *Let (x, y) be in $\mathfrak{g} \times \mathfrak{g}$ such that $P_{x,y} \cap \mathfrak{g}_{\text{reg}}$ is not empty.*

- (i) *Let O be an open subset of \mathbb{K}^2 such that $ax + by$ is in $\mathfrak{g}_{\text{reg}}$ for all (a, b) in O . Then $V_{x,y}$ is the sum of the \mathfrak{g}^{ax+by} , $(a, b) \in O$.*
- (ii) *The spaces $[x, V_{x,y}]$ and $[y, V_{x,y}]$ are equal.*
- (iii) *The space $[x, V_{x,y}]$ is orthogonal to $V_{x,y}$. Furthermore, (x, y) is in $\Omega_{\mathfrak{g}}$ if and only if $[x, V_{x,y}]$ is the orthogonal complement of $V_{x,y}$ in \mathfrak{g} .*
- (iv) *The space $V_{x,y}$ is contained in $V'_{x,y}$. Moreover, $V_{x,y} = V'_{x,y}$ if (x, y) is in $\Omega_{\mathfrak{g}}$.*

- (v) The element (x, y) of $\mathfrak{g} \times \mathfrak{g}$ is in $\Omega_{\mathfrak{g}}$ if and only if $V_{x,y}$ has dimension $b_{\mathfrak{g}}$.
- (vi) For g in G , for $i = 1, \dots, \ell$ and for $m = 0, \dots, d_i - 1$, $\varepsilon_i^{(m)}$ is a G -equivariant map.

Proof. (i) For pairwise different elements $t_1, \dots, t_{d_\ell-1}$ of $\mathbb{k} \setminus \{0\}$, the $\varepsilon_i^{(m)}(x, y)$'s, $m = 0, \dots, d_i - 1$ are linear combinations of the $\varepsilon_i(x + t_j y)$'s, $j = 1, \dots, d_i - 1$ for $i = 1, \dots, \ell$. Furthermore, for all z in $\mathfrak{g}_{\text{reg}}$, $\varepsilon_1(z), \dots, \varepsilon_\ell(z)$ is a basis of \mathfrak{g}^z by [Ko63, Theorem 9], whence the assertion since the maps $\varepsilon_1, \dots, \varepsilon_\ell$ are homogeneous.

(ii) Let O be an open subset of $(\mathbb{k} \setminus \{0\})^2$ such that $ax + by$ is in $\mathfrak{g}_{\text{reg}}$ for all (a, b) in O . For all (a, b) in O , $[x, \mathfrak{g}^{ax+by}] = [y, \mathfrak{g}^{ax+by}]$ since $[ax + by, \mathfrak{g}^{ax+by}] = 0$ and since $ab \neq 0$, whence the assertion by (i).

(iii) results from [Bol91, Theorem 2.1].

(iv) According to [Ko63, Theorem 9], for all z in \mathfrak{g} and for $i = 1, \dots, \ell$, $\varepsilon_i(z)$ is in \mathfrak{g}^z . Hence for all t in \mathbb{k} , $\varepsilon_i(x + ty)$ is in $V'_{x,y}$. So $\varepsilon_i^{(m)}(x, y)$ is in $V'_{x,y}$ for all m , whence $V_{x,y} \subset V'_{x,y}$.

Let suppose that (x, y) is in $\Omega_{\mathfrak{g}}$. According to [Ko63, Theorem 9], for all (a, b) in $\mathbb{k}^2 \setminus \{0\}$, $\varepsilon_1(ax + by), \dots, \varepsilon_\ell(ax + by)$ is a basis of \mathfrak{g}^{ax+by} . Hence \mathfrak{g}^{ax+by} is contained in $V_{x,y}$, whence the assertion.

(v) Since $V_{x,y}$ only depends on $P_{x,y}$, one can suppose x regular. Then \mathfrak{g}^x is contained in $V_{x,y}$ by [Ko63, Theorem 9] and $[x, V_{x,y}]$ has dimension $\dim V_{x,y} - \ell$. As a result, by (iii),

$$2\dim V_{x,y} - \ell \leq \dim \mathfrak{g}$$

and the equality holds if and only if $[x, V_{x,y}]$ is the orthogonal complement of $V_{x,y}$ in \mathfrak{g} , whence the assertion by (iii) again.

(vi) Let i be in $\{1, \dots, \ell\}$. Since p_i is G -invariant, ε_i is a G -equivariant map. As a result, its 2-polarizations $\varepsilon_i^{(0)}, \dots, \varepsilon_i^{(d_i-1)}$ are G -equivariant for the diagonal action of G in $\mathfrak{g} \times \mathfrak{g}$. \square

Theorem 2.2. (i) The module $B_{\mathfrak{g}}$ is a free module of rank $b_{\mathfrak{g}}$ whose a basis is the sequence $\varepsilon_i^{(0)}, \dots, \varepsilon_i^{(d_i-1)}$, $i = 1, \dots, \ell$.

(ii) For φ in $S(\mathfrak{g} \times \mathfrak{g}) \otimes_{\mathbb{k}} \mathfrak{g}$, φ is in $B_{\mathfrak{g}}$ if and only if $p\varphi \in B_{\mathfrak{g}}$ for some p in $S(\mathfrak{g} \times \mathfrak{g}) \setminus \{0\}$.

(iii) For all φ in $B_{\mathfrak{g}}$ and for all (x, y) in $\mathfrak{g} \times \mathfrak{g}$, $\varphi(x, y)$ is orthogonal to $[x, y]$.

Proof. (i) and (ii) According to Proposition 2.1, (iv), $\varepsilon_i^{(m)}$ is in $B_{\mathfrak{g}}$ for all (i, m) . Moreover, according to Proposition 2.1, (v), these elements are linearly independent over $S(\mathfrak{g} \times \mathfrak{g})$ since the sum of the degrees d_1, \dots, d_ℓ equal $b_{\mathfrak{g}}$ by [Bou02, Ch. V, §5, Proposition 3]. Let φ be an element of $S(\mathfrak{g} \times \mathfrak{g}) \otimes_{\mathbb{k}} \mathfrak{g}$ such that $p\varphi$ is in $B_{\mathfrak{g}}$ for some p in $S(\mathfrak{g} \times \mathfrak{g}) \setminus \{0\}$. Since $\Omega_{\mathfrak{g}}$ is a big open subset of $\mathfrak{g} \times \mathfrak{g}$, for all (x, y) in a dense open subset of $\Omega_{\mathfrak{g}}$, $\varphi(x, y)$ is in $V_{x,y}$ by Proposition 2.1, (iv). According to Proposition 2.1, (v), the map

$$\Omega_{\mathfrak{g}} \longrightarrow \text{Gr}_{b_{\mathfrak{g}}}(\mathfrak{g}) \quad (x, y) \longmapsto V_{x,y}$$

is regular. So, $\varphi(x, y)$ is in $V_{x,y}$ for all (x, y) in $\Omega_{\mathfrak{g}}$ and for some regular functions $a_{i,m}$, $i = 1, \dots, \ell$, $m = 0, \dots, d_i - 1$ on $\Omega_{\mathfrak{g}}$,

$$\varphi(x, y) = \sum_{i=1}^{\ell} \sum_{m=0}^{d_i-1} a_{i,m}(x, y) \varepsilon_i^{(m)}(x, y)$$

for all (x, y) in $\Omega_{\mathfrak{g}}$. Since $\Omega_{\mathfrak{g}}$ is a big open subset of $\mathfrak{g} \times \mathfrak{g}$ and since $\mathfrak{g} \times \mathfrak{g}$ is normal, the $a_{i,m}$'s have a regular extension to $\mathfrak{g} \times \mathfrak{g}$. Hence φ is a linear combination of the $\varepsilon_i^{(m)}$'s with coefficients in $S(\mathfrak{g} \times \mathfrak{g})$. As

a result, the sequence $\varepsilon_i^{(m)}$, $i = 1, \dots, \ell$, $m = 0, \dots, d_i - 1$ is a basis of the module $B_{\mathfrak{g}}$ and $B_{\mathfrak{g}}$ is the subset of elements φ of $S(\mathfrak{g} \times \mathfrak{g}) \otimes_{\mathbb{K}} \mathfrak{g}$ such that $p\varphi \in B_{\mathfrak{g}}$ for some p in $S(\mathfrak{g} \times \mathfrak{g}) \setminus \{0\}$.

(iii) Let φ be in $B_{\mathfrak{g}}$. According to (i) and Proposition 2.1,(iii) and (iv), for all (x, y) in $\Omega_{\mathfrak{g}}$, $[x, \varphi(x, y)]$ is orthogonal to $V_{x,y}$. Then, since y is in $V_{x,y}$, $[x, \varphi(x, y)]$ is orthogonal to y and $\langle \varphi(x, y), [x, y] \rangle = 0$, whence the assertion. \square

2.2. Let also denote by $\langle \cdot, \cdot \rangle$ the natural extension of $\langle \cdot, \cdot \rangle$ to the module $S(\mathfrak{g} \times \mathfrak{g}) \otimes_{\mathbb{K}} \mathfrak{g}$.

Proposition 2.3. *Let $C_{\mathfrak{g}}$ be the orthogonal complement of $B_{\mathfrak{g}}$ in $S(\mathfrak{g} \times \mathfrak{g}) \otimes_{\mathbb{K}} \mathfrak{g}$.*

(i) *For φ in $S(\mathfrak{g} \times \mathfrak{g}) \otimes_{\mathbb{K}} \mathfrak{g}$, φ is in $C_{\mathfrak{g}}$ if and only if $\varphi(x, y)$ is in $[x, V_{x,y}]$ for all (x, y) in a nonempty open subset of $\mathfrak{g} \times \mathfrak{g}$.*

(ii) *The module $C_{\mathfrak{g}}$ is free of rank $b_{\mathfrak{g}} - \ell$. Furthermore, the sequence of maps*

$$(x, y) \mapsto [x, \varepsilon_i^{(1)}(x, y)], \dots, (x, y) \mapsto [x, \varepsilon_i^{(d_i-1)}(x, y)], \quad i = 1, \dots, \ell$$

is a basis of $C_{\mathfrak{g}}$.

(iii) *The orthogonal complement of $C_{\mathfrak{g}}$ in $S(\mathfrak{g} \times \mathfrak{g}) \otimes_{\mathbb{K}} \mathfrak{g}$ equals $B_{\mathfrak{g}}$.*

Proof. (i) Let φ be in $S(\mathfrak{g} \times \mathfrak{g}) \otimes_{\mathbb{K}} \mathfrak{g}$. If φ is in $C_{\mathfrak{g}}$, then $\varphi(x, y)$ is orthogonal to $V_{x,y}$ for all (x, y) in $\Omega_{\mathfrak{g}}$. Then, according to Proposition 2.1,(iii), $\varphi(x, y)$ is in $[x, V_{x,y}]$ for all (x, y) in $\Omega_{\mathfrak{g}}$. Conversely, let suppose that $\varphi(x, y)$ is in $[x, V_{x,y}]$ for all (x, y) in a nonempty open subset V of $\mathfrak{g} \times \mathfrak{g}$. By Proposition 2.1,(iii) again, for all (x, y) in $V \cap \Omega_{\mathfrak{g}}$, $\varphi(x, y)$ is orthogonal to the $\varepsilon_i^{(m)}(x, y)$'s, $i = 1, \dots, \ell$, $m = 0, \dots, d_i - 1$, whence the assertion by Theorem 2.1.

(ii) Let C be the submodule of $S(\mathfrak{g} \times \mathfrak{g}) \otimes_{\mathbb{K}} \mathfrak{g}$ generated by the maps

$$(x, y) \mapsto [x, \varepsilon_i^{(1)}(x, y)], \dots, (x, y) \mapsto [x, \varepsilon_i^{(d_i-1)}(x, y)], \quad i = 1, \dots, \ell$$

According to (i), C is a submodule of $C_{\mathfrak{g}}$. This module is free of rank $b_{\mathfrak{g}} - \ell$ since $[x, V_{x,y}]$ has dimension $b_{\mathfrak{g}} - \ell$ for all (x, y) in $\Omega_{\mathfrak{g}}$ by Proposition 2.1, (v). According to (i), for φ in $C_{\mathfrak{g}}$, for all (x, y) in $\Omega_{\mathfrak{g}}$,

$$\varphi(x, y) = \sum_{i=1}^{\ell} \sum_{m=1}^{d_i-1} a_{i,m}(x, y) [x, \varepsilon_i^{(m)}(x, y)]$$

with the $a_{i,m}$'s regular on $\Omega_{\mathfrak{g}}$ and uniquely defined by this equality. Since $\Omega_{\mathfrak{g}}$ is a big open subset of $\mathfrak{g} \times \mathfrak{g}$ and since $\mathfrak{g} \times \mathfrak{g}$ is normal, the $a_{i,m}$'s have a regular extension to $\mathfrak{g} \times \mathfrak{g}$. As a result, φ is in C , whence the assertion.

(iii) Let φ be in the orthogonal complement of $C_{\mathfrak{g}}$ in $S(\mathfrak{g} \times \mathfrak{g}) \otimes_{\mathbb{K}} \mathfrak{g}$. According to (ii), for all (x, y) in $\Omega_{\mathfrak{g}}$, $\varphi(x, y)$ is orthogonal to $[x, V_{x,y}]$. Hence by Proposition 2.1,(iii), $\varphi(x, y)$ is in $V_{x,y}$ for all (x, y) in $\Omega_{\mathfrak{g}}$. So, by Theorem 2.1, φ is in $B_{\mathfrak{g}}$, whence the assertion. \square

Let denote by \mathcal{B} and \mathcal{C} the localizations of $B_{\mathfrak{g}}$ and $C_{\mathfrak{g}}$ on $\mathfrak{g} \times \mathfrak{g}$ respectively. For (x, y) in $\mathfrak{g} \times \mathfrak{g}$, let $C_{x,y}$ be the image of $C_{\mathfrak{g}}$ by the evaluation map at (x, y) .

Lemma 2.4. *There exists an affine open cover \mathcal{O} of $\Omega_{\mathfrak{g}}$ verifying the following condition: for all O in \mathcal{O} , there exist some subspaces E and F of \mathfrak{g} , depending on O , such that*

$$\mathfrak{g} = E \oplus V_{x,y} = F \oplus C_{x,y}$$

for all (x, y) in O . Moreover, for all (x, y) in O , the orthogonal complement of $V_{x,y}$ in \mathfrak{g} equals $C_{x,y}$.

Proof. According to Proposition 2.1,(iii) and (v), for all (x, y) in Ω_g , $V_{x,y}$ and $C_{x,y}$ have dimension b_g and $b_g - \ell$ respectively so that the maps

$$\Omega_g \longrightarrow \text{Gr}_{b_g}(g) \quad (x, y) \longmapsto V_{x,y} \quad \Omega_g \longrightarrow \text{Gr}_{b_g-\ell}(g) \quad (x, y) \longmapsto C_{x,y}$$

are regular, whence the assertion. \square

3. TORSION AND PROJECTIVE DIMENSION

Let E and $E^\#$ be the quotients of $S(g \times g) \otimes_{\mathbb{K}} g$ by B_g and C_g respectively. For i positive integer, let denote by E_i the quotient of $\bigwedge^i(E)$ by its torsion module.

3.1. Let B_g^* and C_g^* be the duals of B_g and C_g .

Lemma 3.1. (i) *The $S(g \times g)$ -modules E and $E^\#$ have projective dimension at most 1.*

(ii) *The $S(g \times g)$ -modules E and $E^\#$ are torsion free.*

(iii) *The modules C_g and B_g are the duals of E and $E^\#$ respectively.*

(iv) *The canonical morphisms from E to C_g^* is an embedding.*

Proof. (i) By definition, the short sequences of $S(g \times g)$ -modules,

$$0 \longrightarrow B_g \longrightarrow S(g \times g) \otimes_{\mathbb{K}} g \longrightarrow E \longrightarrow 0$$

$$0 \longrightarrow C_g \longrightarrow S(g \times g) \otimes_{\mathbb{K}} g \longrightarrow E^\# \longrightarrow 0$$

are exact. Hence E and $E^\#$ have projective dimension at most 1 since B_g and C_g are free modules by Theorem 2.2 and Proposition 2.3,(ii).

(ii) The module E is torsion free by Theorem 2.2,(ii). By definition, for φ in $S(g \times g) \otimes_{\mathbb{K}} g$, φ is in C_g if $p\varphi$ is in C_g for some p in $S(g \times g) \setminus \{0\}$, whence $E^\#$ is torsion free.

(iii) According to the exact sequences of (i), the dual of E is the orthogonal complement of B_g in $S(g \times g) \otimes_{\mathbb{K}} g$ and the dual of $E^\#$ is the orthogonal complement of C_g in $S(g \times g) \otimes_{\mathbb{K}} g$, whence the assertion since C_g is the orthogonal complement of B_g in $S(g \times g) \otimes_{\mathbb{K}} g$ by definition and since B_g is the orthogonal complement of C_g in $S(g \times g) \otimes_{\mathbb{K}} g$ by Proposition 2.3,(iii).

(iv) Let $\bar{\omega}$ be in the kernel of the canonical morphism from E to C_g^* . Let ω be a representative of $\bar{\omega}$ in $S(g \times g) \otimes_{\mathbb{K}} g$. According to Proposition 2.3,(iii), B_g is the orthogonal complement of C_g in $S(g \times g) \otimes_{\mathbb{K}} g$ so that ω is in B_g , whence the assertion. \square

Let ι be the morphism

$$S(g \times g) \otimes_{\mathbb{K}} g \longrightarrow C_g^* \quad v \longmapsto (\mu \mapsto \langle v, \mu \rangle)$$

Lemma 3.2. *The submodule $\iota(S(g \times g) \otimes_{\mathbb{K}} g)$ of C_g^* has projective dimension at most 1.*

Proof. According to Proposition 2.3,(iii), B_g is the kernel of ι , whence the assertion since B_g is a free module by Theorem 2.2. \square

Let set

$$\mathcal{E} = \bigwedge_{i=1}^{\ell} \mathcal{E}_i^{(0)} \wedge \cdots \wedge \mathcal{E}_i^{(d_i-1)}$$

and for i positive integer, let denote by θ_i the morphism

$$S(\mathfrak{g} \times \mathfrak{g}) \otimes_{\mathbb{K}} \wedge^i(\mathfrak{g}) \longrightarrow \wedge^i(\mathfrak{g}) \wedge \wedge^{b_{\mathfrak{g}}}(\mathbf{B}_{\mathfrak{g}}) \quad \varphi \longmapsto \varphi \wedge \varepsilon$$

Proposition 3.3. *Let i be a positive integer.*

- (i) *The morphism θ_i defines through the quotient an isomorphism from E_i onto $\wedge^i(\mathfrak{g}) \wedge \wedge^{b_{\mathfrak{g}}}(\mathbf{B}_{\mathfrak{g}})$.*
- (ii) *The short sequence of $S(\mathfrak{g} \times \mathfrak{g})$ -modules*

$$0 \longrightarrow \mathbf{B}_{\mathfrak{g}} \otimes_{S(\mathfrak{g} \times \mathfrak{g})} E_i \longrightarrow \mathfrak{g} \otimes_{\mathbb{K}} E_i \longrightarrow E \otimes_{S(\mathfrak{g} \times \mathfrak{g})} E_i \longrightarrow 0$$

is exact.

Proof. (i) For j positive integer, let denote by π_j the canonical map from $S(\mathfrak{g} \times \mathfrak{g}) \otimes_{\mathbb{K}} \wedge^j(\mathfrak{g})$ to $\wedge^j(E)$. Let ω be in the kernel of π_i . Let O be an element of the affine open cover of $\Omega_{\mathfrak{g}}$ of Lemma 2.4 and let W be a subspace of \mathfrak{g} such that

$$\mathfrak{g} = W \oplus V_{x,y}$$

for all (x, y) in O so that π_1 induces an isomorphism

$$\mathbb{K}[O] \otimes_{\mathbb{K}} W \longrightarrow \mathbb{K}[O] \otimes_{S(\mathfrak{g} \times \mathfrak{g})} E$$

Moreover, $\mathbf{B}_{\mathfrak{g}}$ is the kernel of π_1 . Then, from the equality

$$\mathbb{K}[O] \otimes_{\mathbb{K}} \wedge^i(\mathfrak{g}) = \bigoplus_{j=0}^i \wedge^j(W) \wedge \mathbb{K}[O] \otimes_{S(\mathfrak{g} \times \mathfrak{g})} \wedge^{i-j}(\mathbf{B}_{\mathfrak{g}})$$

it results that the restriction of ω to O is in $\mathbb{K}[O] \otimes_{S(\mathfrak{g} \times \mathfrak{g})} \wedge^{i-1}(\mathfrak{g}) \wedge \mathbf{B}_{\mathfrak{g}}$. Hence the restriction of $\omega \wedge \varepsilon$ to O equals 0 and ω is in the kernel of θ_i since $\wedge^i(\mathfrak{g}) \wedge \wedge^{b_{\mathfrak{g}}}(\mathbf{B}_{\mathfrak{g}})$ has no torsion as a submodule of a free module. As a result, θ_i defines through the quotient a morphism from $\wedge^i(E)$ to $\wedge^i(\mathfrak{g}) \wedge \wedge^{b_{\mathfrak{g}}}(\mathbf{B}_{\mathfrak{g}})$. Let denote it by ϑ'_i . Since $\wedge^i(\mathfrak{g}) \wedge \wedge^{b_{\mathfrak{g}}}(\mathbf{B}_{\mathfrak{g}})$ is torsion free, the torsion submodule of $\wedge^i(E)$ is contained in the kernel of ϑ'_i . Hence ϑ'_i defines through the quotient a morphism from E_i to $\wedge^i(\mathfrak{g}) \wedge \wedge^{b_{\mathfrak{g}}}(\mathbf{B}_{\mathfrak{g}})$. Denoting it by ϑ_i , ϑ'_i and ϑ_i are surjective since θ_i is too.

Let $\overline{\omega}$ be in the kernel of ϑ'_i and let ω be a representative of $\overline{\omega}$ in $S(\mathfrak{g} \times \mathfrak{g}) \otimes_{\mathbb{K}} \wedge^i(\mathfrak{g})$. Then $\omega \wedge \varepsilon = 0$ so that the restriction of ω to the above open subset O is in $\mathbb{K}[O] \otimes_{S(\mathfrak{g} \times \mathfrak{g})} \wedge^{i-1}(\mathfrak{g}) \wedge \mathbf{B}_{\mathfrak{g}}$. As a result, the restriction of $\overline{\omega}$ to O equals 0. So, $\overline{\omega}$ is in the torsion submodule of $\wedge^i(E)$, whence the assertion.

- (ii) By definition, the sequence

$$0 \longrightarrow \mathbf{B}_{\mathfrak{g}} \longrightarrow S(\mathfrak{g} \times \mathfrak{g}) \otimes_{\mathbb{K}} \mathfrak{g} \longrightarrow E \longrightarrow 0$$

is exact. Then the sequence

$$\mathrm{Tor}_1^{S(\mathfrak{g} \times \mathfrak{g})}(E, E_i) \longrightarrow \mathbf{B}_{\mathfrak{g}} \otimes_{S(\mathfrak{g} \times \mathfrak{g})} E_i \longrightarrow \mathfrak{g} \otimes_{\mathbb{K}} E_i \longrightarrow E \otimes_{S(\mathfrak{g} \times \mathfrak{g})} E_i \longrightarrow 0$$

is exact. By definition, E_i is torsion free. As a result, $\mathbf{B}_{\mathfrak{g}} \otimes_{S(\mathfrak{g} \times \mathfrak{g})} E_i$ is torsion free since $\mathbf{B}_{\mathfrak{g}}$ is a free module. Then, since $\mathrm{Tor}_1^{S(\mathfrak{g} \times \mathfrak{g})}(E, E_i)$ is a torsion module, its image in $\mathbf{B}_{\mathfrak{g}} \otimes_{S(\mathfrak{g} \times \mathfrak{g})} E_i$ equals 0, whence the assertion. \square

3.2. For i positive integer, $\langle \cdot, \cdot \rangle$ has a canonical extension to $S(\mathfrak{g} \times \mathfrak{g}) \otimes_{\mathbb{K}} \wedge^i(\mathfrak{g})$ denoted again by $\langle \cdot, \cdot \rangle$.

Lemma 3.4. *Let i be a positive integer. Let T_i be the torsion module of $E \otimes_{S(\mathfrak{g} \times \mathfrak{g})} E_i$ and let T'_i be its inverse image by the canonical morphism $\mathfrak{g} \otimes_{\mathbb{K}} E_i \rightarrow E \otimes_{S(\mathfrak{g} \times \mathfrak{g})} E_i$.*

(i) *The canonical morphism from $\wedge^i(E)$ to $\wedge^i(C_{\mathfrak{g}}^*)$ defines through the quotient an embedding of E_i into $\wedge^i(C_{\mathfrak{g}}^*)$.*

(ii) *The module of T'_i is the intersection of $\mathfrak{g} \otimes_{\mathbb{K}} E_i$ and $B_{\mathfrak{g}} \otimes_{S(\mathfrak{g} \times \mathfrak{g})} \wedge^i(C_{\mathfrak{g}}^*)$.*

(iii) *The module T'_i is isomorphic to $\text{Hom}_{S(\mathfrak{g} \times \mathfrak{g})}(E^{\#}, E_i)$.*

Proof. (i) According to Lemma 3.1.(iii), there is a canonical morphism from $\wedge^i(E)$ to $\wedge^i(C_{\mathfrak{g}}^*)$. Let $\bar{\omega}$ be in its kernel and let ω be a representative of $\bar{\omega}$ in $S(\mathfrak{g} \times \mathfrak{g}) \otimes_{\mathbb{K}} \wedge^i(\mathfrak{g})$. Then ω is orthogonal to $\wedge^i(C_{\mathfrak{g}})$ with respect to $\langle \cdot, \cdot \rangle$. So for O as in Lemma 2.4, the restriction of ω to O is in $\mathbb{K}[O] \otimes_{S(\mathfrak{g} \times \mathfrak{g})} \wedge^{i-1}(\mathfrak{g}) \wedge B_{\mathfrak{g}}$. Hence the restriction of $\bar{\omega}$ to O equals 0. In other words, $\bar{\omega}$ is in the torsion module of $\wedge^i(E)$, whence the assertion since $\wedge^i(C_{\mathfrak{g}}^*)$ is a free module.

(ii) Since $\wedge^i(C_{\mathfrak{g}}^*)$ is a free module, by Proposition 3.3.(ii), there is a morphism of short exact sequences

$$\begin{array}{ccccccc} 0 & \longrightarrow & B_{\mathfrak{g}} \otimes_{S(\mathfrak{g} \times \mathfrak{g})} E_i & \longrightarrow & \mathfrak{g} \otimes_{\mathbb{K}} E_i & \longrightarrow & E \otimes_{S(\mathfrak{g} \times \mathfrak{g})} E_i \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & B_{\mathfrak{g}} \otimes_{S(\mathfrak{g} \times \mathfrak{g})} \wedge^i(C_{\mathfrak{g}}^*) & \longrightarrow & \mathfrak{g} \otimes_{\mathbb{K}} \wedge^i(C_{\mathfrak{g}}^*) & \longrightarrow & E \otimes_{S(\mathfrak{g} \times \mathfrak{g})} \wedge^i(C_{\mathfrak{g}}^*) \longrightarrow 0 \end{array}$$

Moreover, the tow first vertical arrows are embeddings. Hence T'_i is the intersection of $\mathfrak{g} \otimes_{\mathbb{K}} E_i$ and $B_{\mathfrak{g}} \otimes_{S(\mathfrak{g} \times \mathfrak{g})} \wedge^i(C_{\mathfrak{g}}^*)$.

(iii) According to the identification of \mathfrak{g} with its dual, $\mathfrak{g} \otimes_{\mathbb{K}} E_i = \text{Hom}_{\mathbb{K}}(\mathfrak{g}, E_i)$. Moreover, according to the short exact sequence of $S(\mathfrak{g} \times \mathfrak{g})$ -modules

$$0 \longrightarrow C_{\mathfrak{g}} \longrightarrow S(\mathfrak{g} \times \mathfrak{g}) \otimes_{\mathbb{K}} \mathfrak{g} \longrightarrow E^{\#} \longrightarrow 0$$

the sequence of $S(\mathfrak{g} \times \mathfrak{g})$ -modules

$$0 \longrightarrow \text{Hom}_{S(\mathfrak{g} \times \mathfrak{g})}(E^{\#}, E_i) \longrightarrow \text{Hom}_{\mathbb{K}}(\mathfrak{g}, E_i) \longrightarrow \text{Hom}_{S(\mathfrak{g} \times \mathfrak{g})}(C_{\mathfrak{g}}, E_i) \longrightarrow \text{Ext}_{S(\mathfrak{g} \times \mathfrak{g})}^1(E^{\#}, E_i)$$

is exact. For φ in $\text{Hom}_{\mathbb{K}}(\mathfrak{g}, E_i)$, φ is in the kernel of the third arrow if and only if $C_{\mathfrak{g}}$ is contained in the kernel of φ or equivalently φ is in $B_{\mathfrak{g}} \otimes_{S(\mathfrak{g} \times \mathfrak{g})} \wedge^i(C_{\mathfrak{g}}^*)$, whence the assertion by (ii). \square

The following corollary results from Lemma 3.4.

Corollary 3.5. *Let i be a positive integer and let \overline{E}_i be the quotient of $E \otimes_{S(\mathfrak{g} \times \mathfrak{g})} E_i$ by its torsion module. Then the short sequence of $S(\mathfrak{g} \times \mathfrak{g})$ -modules*

$$0 \longrightarrow \text{Hom}_{S(\mathfrak{g} \times \mathfrak{g})}(E^{\#}, E_i) \longrightarrow \mathfrak{g} \otimes_{\mathbb{K}} E_i \longrightarrow \overline{E}_i \longrightarrow 0$$

is exact.

3.3. For i positive integer, let d_i and d'_i be the projective dimensions of E_i and $\text{Hom}_{S(\mathfrak{g} \times \mathfrak{g})}(E^\#, E_i)$.

Lemma 3.6. *Let i be a positive integer and let d''_i be the projective dimension of $\text{Ext}^1_{S(\mathfrak{g} \times \mathfrak{g})}(E^\#, E_i)$.*

(i) *The integer d'_i is at most $\sup\{d''_i - 2, d_i\}$.*

(ii) *For P a projective module over $S(\mathfrak{g} \times \mathfrak{g})$, $\text{Ext}^1_{S(\mathfrak{g} \times \mathfrak{g})}(E^\#, P)$ is canonically isomorphic to $A_{\mathfrak{g}} \otimes_{S(\mathfrak{g} \times \mathfrak{g})} P$ with $A_{\mathfrak{g}}$ the quotient of $C_{\mathfrak{g}}^*$ by $\iota(S(\mathfrak{g} \times \mathfrak{g}) \otimes_{\mathbb{K}} \mathfrak{g})$.*

(iii) *The integer d''_i is at most $d_i + 2$.*

Proof. (i) From the short exact sequence

$$0 \longrightarrow C_{\mathfrak{g}} \longrightarrow S(\mathfrak{g} \times \mathfrak{g}) \otimes_{\mathbb{K}} \mathfrak{g} \longrightarrow E^\# \longrightarrow 0$$

one deduces the exact sequence

$$0 \longrightarrow \text{Hom}_{S(\mathfrak{g} \times \mathfrak{g})}(E^\#, E_i) \longrightarrow \text{Hom}_{\mathbb{K}}(\mathfrak{g}, E_i) \longrightarrow \text{Hom}_{S(\mathfrak{g} \times \mathfrak{g})}(C_{\mathfrak{g}}, E_i) \longrightarrow \text{Ext}^1_{S(\mathfrak{g} \times \mathfrak{g})}(E^\#, E_i) \longrightarrow 0$$

whence the two short exact sequences

$$0 \longrightarrow \text{Hom}_{S(\mathfrak{g} \times \mathfrak{g})}(E^\#, E_i) \longrightarrow \text{Hom}_{\mathbb{K}}(\mathfrak{g}, E_i) \longrightarrow Z \longrightarrow 0$$

$$0 \longrightarrow Z \longrightarrow \text{Hom}_{S(\mathfrak{g} \times \mathfrak{g})}(C_{\mathfrak{g}}, E_i) \longrightarrow \text{Ext}^1_{S(\mathfrak{g} \times \mathfrak{g})}(E^\#, E_i) \longrightarrow 0$$

with Z the image of the arrow

$$\text{Hom}_{\mathbb{K}}(\mathfrak{g}, E_i) \longrightarrow \text{Hom}_{S(\mathfrak{g} \times \mathfrak{g})}(C_{\mathfrak{g}}, E_i)$$

Denoting by d the projective dimension of Z , one deduces the inequalities

$$d'_i \leq \sup\{d - 1, d_i\} \quad d \leq \sup\{d''_i - 1, d_i\}$$

since $C_{\mathfrak{g}}$ is a free module, whence the assertion.

(ii) From the short exact sequence

$$0 \longrightarrow C_{\mathfrak{g}} \longrightarrow S(\mathfrak{g} \times \mathfrak{g}) \otimes_{\mathbb{K}} \mathfrak{g} \longrightarrow E^\# \longrightarrow 0$$

one deduces the exact sequence

$$0 \longrightarrow \text{Hom}_{S(\mathfrak{g} \times \mathfrak{g})}(E^\#, P) \longrightarrow \text{Hom}_{\mathbb{K}}(\mathfrak{g}, P) \longrightarrow \text{Hom}_{S(\mathfrak{g} \times \mathfrak{g})}(C_{\mathfrak{g}}, P) \longrightarrow \text{Ext}^1_{S(\mathfrak{g} \times \mathfrak{g})}(E^\#, P) \longrightarrow 0$$

Since $C_{\mathfrak{g}}$ is a free module, $\text{Hom}_{S(\mathfrak{g} \times \mathfrak{g})}(C_{\mathfrak{g}}, P)$ is canonically isomorphic to $C_{\mathfrak{g}}^* \otimes_{S(\mathfrak{g} \times \mathfrak{g})} P$ and there is a commutative diagram

$$\begin{array}{ccc} \text{Hom}_{\mathbb{K}}(\mathfrak{g}, P) & \longrightarrow & \text{Hom}_{S(\mathfrak{g} \times \mathfrak{g})}(C_{\mathfrak{g}}, P) \\ \downarrow & & \downarrow \\ \mathfrak{g} \otimes_{\mathbb{K}} P & \xrightarrow{\iota \otimes \text{id}_P} & C_{\mathfrak{g}}^* \otimes_{S(\mathfrak{g} \times \mathfrak{g})} P \end{array}$$

whenever the vertical arrows are isomorphisms, whence the assertion since P is projective.

(iii) Let

$$0 \longrightarrow P_{d_i} \longrightarrow \cdots \longrightarrow P_0 \longrightarrow E_i \longrightarrow 0$$

be a projective resolution of E_i . For j nonnegative integer, let Z_j be the space of cycles of this complex. Since $E^\#$ has projective dimension at most 1, for all module M over $S(\mathfrak{g} \times \mathfrak{g})$, $\text{Ext}_{S(\mathfrak{g} \times \mathfrak{g})}^j(E^\#, M) = 0$ for all integer at least 2, whence the exact sequences

$$\begin{aligned} \text{Ext}_{S(\mathfrak{g} \times \mathfrak{g})}^1(E^\#, Z_0) &\longrightarrow \text{Ext}_{S(\mathfrak{g} \times \mathfrak{g})}^1(E^\#, P_0) \longrightarrow \text{Ext}_{S(\mathfrak{g} \times \mathfrak{g})}^1(E^\#, E_i) \longrightarrow 0 \\ \text{Ext}_{S(\mathfrak{g} \times \mathfrak{g})}^1(E^\#, Z_j) &\longrightarrow \text{Ext}_{S(\mathfrak{g} \times \mathfrak{g})}^1(E^\#, P_j) \longrightarrow \text{Ext}_{S(\mathfrak{g} \times \mathfrak{g})}^1(E^\#, Z_{j-1}) \longrightarrow 0 \end{aligned}$$

for all positive integer j and whence the long exact sequence

$$0 \longrightarrow \text{Ext}_{S(\mathfrak{g} \times \mathfrak{g})}^1(E^\#, P_{d_i}) \longrightarrow \cdots \longrightarrow \text{Ext}_{S(\mathfrak{g} \times \mathfrak{g})}^1(E^\#, P_0) \longrightarrow \text{Ext}_{S(\mathfrak{g} \times \mathfrak{g})}^1(E^\#, E_i) \longrightarrow 0$$

Then by (ii), there is an exact sequence

$$0 \longrightarrow A_{\mathfrak{g}} \otimes_{S(\mathfrak{g} \times \mathfrak{g})} P_{d_i} \longrightarrow \cdots \longrightarrow A_{\mathfrak{g}} \otimes_{S(\mathfrak{g} \times \mathfrak{g})} P_0 \longrightarrow \text{Ext}_{S(\mathfrak{g} \times \mathfrak{g})}^1(E^\#, E_i) \longrightarrow 0$$

According to Lemma 3.2, $A_{\mathfrak{g}}$ has projective dimension at most 2. Hence d_i'' is at most $d_i + 2$. \square

The following corollary results from Lemma 3.6,(i) and (iii) and Corollary 3.5.

Corollary 3.7. *Let i be a positive integer. Then \overline{E}_i has projective dimension at most $d_i + 1$.*

3.4. For i a positive integer and for M a $S(\mathfrak{g} \times \mathfrak{g})$ -module, let consider on $M^{\otimes i}$ the canonical action of the symmetric group \mathfrak{S}_i . For σ in \mathfrak{S}_i , let denote by $\epsilon(\sigma)$ its signature. Let $M_{\text{sign}}^{\otimes i}$ be the submodule of elements a of $M^{\otimes i}$ such that $\sigma.a = \epsilon(\sigma)a$ for all σ in \mathfrak{S}_i and let δ_i be the endomorphism of $M^{\otimes i}$,

$$a \mapsto \delta_i(a) = \frac{1}{i!} \sum_{\sigma \in \mathfrak{S}_i} \epsilon(\sigma) \sigma.a$$

Then δ_i is a projection of $M^{\otimes i}$ onto $M_{\text{sign}}^{\otimes i}$.

For L submodule of $C_{\mathfrak{g}}^*$, let denote by L_i the image of $L^{\otimes i}$ by the canonical map from $L^{\otimes i}$ to $(C_{\mathfrak{g}}^*)^{\otimes i}$ and let set $L_{i,\text{sign}} := L_i \cap (C_{\mathfrak{g}}^*)_{\text{sign}}^{\otimes i}$. Let $\overline{\bigwedge^i(L)}$ be the quotient of $\bigwedge^i(L)$ by its torsion module. For $i \geq 2$, let identify \mathfrak{S}_{i-1} with the stabilizer of i in \mathfrak{S}_i and let denote by $L_{i-1,\text{sign},1}$ the submodule of elements a of L_i such that $\sigma.a = \epsilon(\sigma)a$ for all σ in \mathfrak{S}_{i-1} .

Lemma 3.8. *Let i be a positive integer and let L be a submodule of $C_{\mathfrak{g}}^*$.*

- (i) *The module L_i is the quotient of $L^{\otimes i}$ by its torsion module.*
- (ii) *The module $L_{i,\text{sign}}$ is isomorphic to $\overline{\bigwedge^i(L)}$.*
- (iii) *For $i \geq 2$, the module $L_{i,\text{sign}}$ is a direct factor of $L_{i-1,\text{sign},1}$.*
- (iv) *For $i \geq 2$, the module $L_{i-1,\text{sign},1}$ is isomorphic to the quotient of $\overline{\bigwedge^{i-1}(L)} \otimes_{S(\mathfrak{g} \times \mathfrak{g})} L$ by its torsion module.*

Proof. (i) Let L_1 and L_2 be submodules of a free module F over $S(\mathfrak{g} \times \mathfrak{g})$. From the short exact sequence

$$0 \longrightarrow L_2 \longrightarrow F \longrightarrow F/L_2 \longrightarrow 0$$

one deduces the exact sequence

$$\text{Tor}_{S(\mathfrak{g} \times \mathfrak{g})}^1(L_1, L_2) \longrightarrow L_1 \otimes_{S(\mathfrak{g} \times \mathfrak{g})} L_2 \longrightarrow L_1 \otimes_{S(\mathfrak{g} \times \mathfrak{g})} F \longrightarrow L_1 \otimes_{S(\mathfrak{g} \times \mathfrak{g})} (F/L_2) \longrightarrow 0$$

Since F is free, $L_1 \otimes_{S(\mathfrak{g} \times \mathfrak{g})} F$ is torsion free. Hence the kernel of the second arrow is the torsion module of $L_1 \otimes_{S(\mathfrak{g} \times \mathfrak{g})} L_2$ since $\text{Tor}_{S(\mathfrak{g} \times \mathfrak{g})}^1(L_1, L_2)$ is a torsion module, whence the assertion by induction on i .

(ii) There is a commutative diagram

$$\begin{array}{ccc} L^{\otimes i} & \longrightarrow & (C_{\mathfrak{g}}^*)^{\otimes i} \\ \delta_i \downarrow & & \downarrow \delta_i \\ L_{\text{sign}}^{\otimes i} & \longrightarrow & (C_{\mathfrak{g}}^*)_{\text{sign}}^{\otimes i} \end{array}$$

so that $L_{i,\text{sign}}$ is the image of $L_{\text{sign}}^{\otimes i}$ by the canonical morphism $L^{\otimes i} \longrightarrow (C_{\mathfrak{g}}^*)^{\otimes i}$, whence a commutative diagram

$$\begin{array}{ccc} L_{\text{sign}}^{\otimes i} & \longrightarrow & \bigwedge^i(L) \\ \downarrow & & \downarrow \\ (C_{\mathfrak{g}}^*)_{\text{sign}}^{\otimes i} & \longrightarrow & \bigwedge^i(C_{\mathfrak{g}}^*) \end{array}$$

According to (i), the kernel of the left down arrow is the torsion module of $L_{\text{sign}}^{\otimes i}$ so that the kernel of the right down arrow is the torsion module of $\bigwedge^i(L)$ since the horizontal arrows are isomorphisms. Moreover, the image of $L_{i,\text{sign}}$ in $\bigwedge^i(C_{\mathfrak{g}}^*)$ is the image of $\bigwedge^i(L)$. Hence $\bigwedge^i(L)$ is isomorphic to $L_{i,\text{sign}}$.

(iii) Let denote by Q_i the kernel of the endomorphism δ_i of $(C_{\mathfrak{g}}^*)^{\otimes i}$. Since δ_i is a projection onto $(C_{\mathfrak{g}}^*)_{\text{sign}}^{\otimes i}$ such that $\delta_i(L_i)$ is contained in $L_{i,\text{sign}}$,

$$(C_{\mathfrak{g}}^*)^{\otimes i} = (C_{\mathfrak{g}}^*)_{\text{sign}}^{\otimes i} \oplus Q_i \quad L_i = L_{i,\text{sign}} \oplus Q_i \cap L_i$$

whence

$$L_{i-1,\text{sign},1} = L_{i,\text{sign}} \oplus Q_i \cap L_{i-1,\text{sign},1}$$

since $L_{i,\text{sign}}$ is a submodule of $L_{i-1,\text{sign},1}$.

(iv) Let L'_i be the image of $L_{i-1,\text{sign},1}$ by the canonical morphism $(C_{\mathfrak{g}}^*)^{\otimes i} \rightarrow \bigwedge^{i-1}(C_{\mathfrak{g}}^*) \otimes_{S(\mathfrak{g} \times \mathfrak{g})} C_{\mathfrak{g}}^*$. Then L'_i is contained in $\bigwedge^{i-1}(C_{\mathfrak{g}}^*) \otimes_{S(\mathfrak{g} \times \mathfrak{g})} L$ since $\bigwedge^{i-1}(C_{\mathfrak{g}}^*) \otimes_{S(\mathfrak{g} \times \mathfrak{g})} L$ is a submodule of $\bigwedge^{i-1}(C_{\mathfrak{g}}^*) \otimes_{S(\mathfrak{g} \times \mathfrak{g})} C_{\mathfrak{g}}^*$. Moreover, the morphism $L_{i-1,\text{sign},1} \rightarrow L'_i$ is an isomorphism since the morphism

$$(C_{\mathfrak{g}}^*)_{\text{sign}}^{\otimes(i-1)} \otimes_{S(\mathfrak{g} \times \mathfrak{g})} C_{\mathfrak{g}}^* \longrightarrow \bigwedge^{i-1}(C_{\mathfrak{g}}^*) \otimes_{S(\mathfrak{g} \times \mathfrak{g})} C_{\mathfrak{g}}^*$$

is too. From (ii), it results the commutative diagram

$$\begin{array}{ccc} L_{\text{sign}}^{\otimes(i-1)} \otimes_{S(\mathfrak{g} \times \mathfrak{g})} L & \longrightarrow & \overline{\bigwedge^{i-1}(L)} \otimes_{S(\mathfrak{g} \times \mathfrak{g})} L \\ \downarrow & & \downarrow \\ L_{i-1,\text{sign},1} & \longrightarrow & L'_i \end{array}$$

with the right down arrow surjective. According to (i), the kernel of the left down arrow is the torsion module of $L_{\text{sign}}^{\otimes(i-1)} \otimes_{S(\mathfrak{g} \times \mathfrak{g})} L$. Hence the kernel of the right down arrow is the torsion module of $\bigwedge^{i-1}(L) \otimes_{S(\mathfrak{g} \times \mathfrak{g})} L$, whence the assertion. \square

Proposition 3.9. *Let i be a positive integer. Then E_i and $\bigwedge^i(\mathfrak{g}) \wedge \bigwedge^{b_{\mathfrak{g}}}(\mathbf{B}_{\mathfrak{g}})$ have projective dimension at most i .*

Proof. According to Proposition 3.3,(i), the modules E_i and $\bigwedge^i(\mathfrak{g}) \wedge \bigwedge^{b_{\mathfrak{g}}}(\mathbf{B}_{\mathfrak{g}})$ are isomorphic. Let prove by induction on i that E_i has projective dimension at most i . By Lemma 3.1,(i), it is true for $i = 1$. Let suppose that it is true for $i - 1$. According to Corollary 3.7, $\overline{E_i}$ has projective dimension at most i . Then by Lemma 3.8, for $L = E$, E_i has projective dimension at most i since E is a submodule of $C_{\mathfrak{g}}^*$ by Lemma 3.1,(iv) and since $E_i = \overline{\bigwedge^i(E)}$ by Lemma 3.4,(i). \square

4. MAIN RESULTS

Let $I_{\mathfrak{g}}$ be the ideal of $S(\mathfrak{g} \times \mathfrak{g})$ generated by the functions $(x, y) \mapsto \langle v, [x, y] \rangle$ with v in \mathfrak{g} . The nullvariety of $I_{\mathfrak{g}}$ in $\mathfrak{g} \times \mathfrak{g}$ is $\mathcal{C}(\mathfrak{g})$.

Lemma 4.1. *Let $J_{\mathfrak{g}}$ be the radical of $I_{\mathfrak{g}}$. Then the support of $J_{\mathfrak{g}}/I_{\mathfrak{g}}$ in $\mathfrak{g} \times \mathfrak{g}$ is strictly contained in $\mathcal{C}(\mathfrak{g})$.*

Proof. Since $\mathcal{C}(\mathfrak{g})$ is the nullvariety of $I_{\mathfrak{g}}$ in $\mathfrak{g} \times \mathfrak{g}$. The support of $J_{\mathfrak{g}}/I_{\mathfrak{g}}$ is contained in $\mathcal{C}(\mathfrak{g})$. Let x_0 be a regular element of \mathfrak{g} and let V be a complement of \mathfrak{g}^{x_0} in \mathfrak{g} . Since the map

$$\mathfrak{g}_{\text{reg}} \longrightarrow \text{Gr}_{\ell}(\mathfrak{g}) \quad x \longmapsto \mathfrak{g}^x$$

is regular, for some open subset O' of $\mathfrak{g}_{\text{reg}}$, containing x_0 ,

$$\mathfrak{g} = V \oplus \mathfrak{g}^x$$

for all x in O' . Let F be a complement of $[x_0, V]$ in \mathfrak{g} . Then the map

$$O' \longrightarrow \text{Gr}_{2n}(\mathfrak{g}) \quad x \longmapsto [x, V]$$

is regular and for some affine open subset O of O' , containing x_0 ,

$$\mathfrak{g} = F \oplus [x, V]$$

for all x in O . Let set:

$$I_O := \mathbb{k}[O \times \mathfrak{g}] \otimes_{S(\mathfrak{g} \times \mathfrak{g})} I_{\mathfrak{g}} \quad J_O := \mathbb{k}[O \times \mathfrak{g}] \otimes_{S(\mathfrak{g} \times \mathfrak{g})} J_{\mathfrak{g}}$$

and let prove $I_O = J_O$. It will result that $\{x_0\} \times \mathfrak{g}^{x_0}$ is not contained in the support of $J_{\mathfrak{g}}/I_{\mathfrak{g}}$.

According to the identification of \mathfrak{g} with its dual,

$$\mathbb{k}[O \times \mathfrak{g}] = \mathbb{k}[O] \otimes_{\mathbb{k}} S(\mathfrak{g})$$

Let denote by v_1, \dots, v_{2n} and f_1, \dots, f_{ℓ} some basis of V and F respectively. For (i, j) in $\mathbb{N}^{2n} \times \mathbb{N}^{\ell}$, let $v_{i,j}$ be the element of $\mathbb{k}[O] \otimes_{\mathbb{k}} S(\mathfrak{g})$,

$$x \longmapsto v_{i,j}(x) := [x, v_1]^{i_1} \cdots [x, v_{2n}]^{i_{2n}} f_1^{j_1} \cdots f_{\ell}^{j_{\ell}}$$

so that $v_{i,j}, (i, j) \in \mathbb{N}^{2n} \times \mathbb{N}^{\ell}$ is a basis of the $\mathbb{k}[O]$ -module $\mathbb{k}[O] \otimes_{\mathbb{k}} S(\mathfrak{g})$. In particular, $v_{i,j}$ is in I_O if

$$|i| := i_1 + \cdots + i_{2n} > 0$$

Let φ be in J_O . For all x in O , $\varphi(x)$ is in the ideal of definition of \mathfrak{g}^x in \mathfrak{g} . Since $[x, \mathfrak{g}]$ is the orthogonal complement of \mathfrak{g}^x in \mathfrak{g} , $\varphi(x)$ is in the ideal of $S(\mathfrak{g})$ generated by $[x, V]$ for all x in O . Hence φ is a linear

combination with coefficients in $\mathbb{k}[O]$ of the $v_{i,j}$'s with $|i| > 0$. As a result, J_O is contained in I_O , whence the lemma. \square

Let d be the $S(\mathfrak{g} \times \mathfrak{g})$ -derivation of the algebra $S(\mathfrak{g} \times \mathfrak{g}) \otimes_{\mathbb{k}} \wedge(\mathfrak{g})$ such that dv is the function $(x, y) \mapsto \langle v, [x, y] \rangle$ on $\mathfrak{g} \times \mathfrak{g}$ for all v in \mathfrak{g} . According to Theorem 2.2, (iii), $\wedge(\mathfrak{g}) \wedge \wedge^{b_{\mathfrak{g}}}(\mathbf{B}_{\mathfrak{g}})$ is a subcomplex of the complex so defined. The gradation on $\wedge(\mathfrak{g})$ induces a gradation on $S(\mathfrak{g} \times \mathfrak{g}) \otimes_{\mathbb{k}} \wedge(\mathfrak{g})$ so that $\wedge(\mathfrak{g}) \wedge \wedge^{b_{\mathfrak{g}}}(\mathbf{B}_{\mathfrak{g}})$ is a graded complex denoted by $C_{\bullet}(\mathfrak{g})$.

Lemma 4.2. *The support of the homology of $C_{\bullet}(\mathfrak{g})$ is contained in $\mathcal{C}(\mathfrak{g})$.*

Proof. Let (x_0, y_0) be in $\mathfrak{g} \times \mathfrak{g} \setminus \mathcal{C}(\mathfrak{g})$ and let v be in \mathfrak{g} such that $\langle v, [x_0, y_0] \rangle \neq 0$. For some affine open subset O of $\mathfrak{g} \times \mathfrak{g}$, containing (x_0, y_0) , $\langle v, [x, y] \rangle \neq 0$ for all (x, y) in O . Then dv is an invertible element of $\mathbb{k}[O]$. For c a cycle of $\mathbb{k}[O] \otimes_{S(\mathfrak{g} \times \mathfrak{g})} C_{\bullet}(\mathfrak{g})$,

$$d(v \wedge c) = (dv)c$$

so that c is a boundary of $\mathbb{k}[O] \otimes_{S(\mathfrak{g} \times \mathfrak{g})} C_{\bullet}(\mathfrak{g})$. \square

Theorem 4.3. (i) *The complex $C_{\bullet}(\mathfrak{g})$ has no homology in degree bigger than $b_{\mathfrak{g}}$.*

(ii) *The ideal $I_{\mathfrak{g}}$ has projective dimension $2n - 1$.*

(iii) *The algebra $S(\mathfrak{g} \times \mathfrak{g})/I_{\mathfrak{g}}$ is Cohen-Macaulay.*

(iv) *The ideal $I_{\mathfrak{g}}$ is prime.*

(v) *The projective dimension of the module $\wedge^n(\mathfrak{g}) \wedge \wedge^{b_{\mathfrak{g}}}(\mathbf{B}_{\mathfrak{g}})$ equals n .*

Proof. (i) Let Z be the space of cycles of degree $b_{\mathfrak{g}} + 1$ of $C_{\bullet}(\mathfrak{g})$. Let set:

$$C_i := \begin{cases} C_{b_{\mathfrak{g}}+1+i}(\mathfrak{g}) & \text{if } i = 1, \dots, n-1 \\ Z & \text{if } i = 0 \end{cases}$$

$$C_{\bullet} := C_0 \oplus \dots \oplus C_{n-1}$$

Then C_{\bullet} is a subcomplex of $C_{\bullet}(\mathfrak{g})$ whose gradation is deduced from the gradation of $C_{\bullet}(\mathfrak{g})$ by translation by $b_{\mathfrak{g}} + 1$. According to Lemma 4.2, the support of its homology is contained in $\mathcal{C}_{\mathfrak{g}}$. In particular, its codimension in $\mathfrak{g} \times \mathfrak{g}$ is

$$4n + 2\ell - (2n + 2\ell) = 2n = n + n - 1 + 1$$

According to Proposition 3.9, for $i = 1, \dots, n-1$, C_i has projective dimension at most i . Hence, by Corollary A.3, C_{\bullet} is acyclic and Z has projective dimension at most $2n - 2$, whence the assertion.

(ii) and (iii) Since $\mathbf{B}_{\mathfrak{g}}$ is a free module of rank $b_{\mathfrak{g}}$, $\wedge^{b_{\mathfrak{g}}}(\mathbf{B}_{\mathfrak{g}})$ is a free module of rank 1. By definition, the short sequence

$$0 \longrightarrow Z \longrightarrow \mathfrak{g} \wedge \wedge^{b_{\mathfrak{g}}}(\mathbf{B}_{\mathfrak{g}}) \longrightarrow I_{\mathfrak{g}} \wedge \wedge^{b_{\mathfrak{g}}}(\mathbf{B}_{\mathfrak{g}}) \longrightarrow 0$$

is exact, whence the short exact sequence

$$0 \longrightarrow Z \longrightarrow \mathfrak{g} \wedge \wedge^{b_{\mathfrak{g}}}(\mathbf{B}_{\mathfrak{g}}) \longrightarrow I_{\mathfrak{g}} \longrightarrow 0$$

Moreover, by Proposition 3.9, $\mathfrak{g} \wedge \wedge^{b_{\mathfrak{g}}}(\mathbf{B}_{\mathfrak{g}})$ has projective dimension at most 1. Then, by (i), $I_{\mathfrak{g}}$ has projective dimension at most $2n - 1$. As a result the $S(\mathfrak{g} \times \mathfrak{g})$ -module $S(\mathfrak{g} \times \mathfrak{g})/I_{\mathfrak{g}}$ has projective dimension

at most $2n$. Then by Auslander-Buchsbaum's theorem [Bou98, §3, Théorème 1], the depth of the graded $S(\mathfrak{g} \times \mathfrak{g})$ -module $S(\mathfrak{g} \times \mathfrak{g})/I_{\mathfrak{g}}$ is at least

$$4b_{\mathfrak{g}} - 2\ell - 2n = 2b_{\mathfrak{g}}$$

so that, according to [Bou98, §1, n°3, Proposition 4], the depth of the graded algebra $S(\mathfrak{g} \times \mathfrak{g})/I_{\mathfrak{g}}$ is at least $2b_{\mathfrak{g}}$. In other words, $S(\mathfrak{g} \times \mathfrak{g})/I_{\mathfrak{g}}$ is Cohen-Macaulay since it has dimension $2b_{\mathfrak{g}}$. Moreover, since the graded algebra $S(\mathfrak{g} \times \mathfrak{g})/I_{\mathfrak{g}}$ has depth $2b_{\mathfrak{g}}$, the graded $S(\mathfrak{g} \times \mathfrak{g})$ -module $S(\mathfrak{g} \times \mathfrak{g})/I_{\mathfrak{g}}$ has depth $2b_{\mathfrak{g}}$ and projective dimension $2n$. Hence $I_{\mathfrak{g}}$ has projective dimension $2n - 1$.

(iii) According to Lemma 4.1, the support in $\mathfrak{g} \times \mathfrak{g}$ of the module $J_{\mathfrak{g}}/I_{\mathfrak{g}}$ is strictly contained in $\mathcal{C}(\mathfrak{g})$. So its codimension in $\mathfrak{g} \times \mathfrak{g}$ is at least $2n + 1$ since $\mathcal{C}(\mathfrak{g})$ is irreducible. Then by (ii) and Proposition A.2.(ii), $I_{\mathfrak{g}} = J_{\mathfrak{g}}$, whence the assertion since $\mathcal{C}(\mathfrak{g})$ is irreducible by [Ri79].

(iv) By (i), $I_{\mathfrak{g}}$ has projective dimension $2n - 1$. Hence, according to Proposition 3.9 and according to (ii) and Corollary A.3, $\bigwedge^n(\mathfrak{g}) \wedge \bigwedge^{b_{\mathfrak{g}}}(\mathbf{B}_{\mathfrak{g}})$ has projective dimension n . \square

Corollary 4.4. *The subscheme of $\mathfrak{g} \times \mathfrak{g}$ defined by $I_{\mathfrak{g}}$ is normal.*

Proof. According to Theorem 4.3.(iii), the subscheme of $\mathfrak{g} \times \mathfrak{g}$ defined by $I_{\mathfrak{g}}$ is Cohen-Macaulay. According to [Po08, Theorem 1], it is smooth in codimension 1. So by Serre's normality criterion [Bou98, §1, n°10, Théorème 4], it is normal. \square

APPENDIX A. PROJECTIVE DIMENSION AND COHOMOLOGY

Let recall in this section classical results. Let X be a Cohen-Macaulay irreducible affine algebraic variety and let S be a closed subset of codimension p of X . Let P_{\bullet} be a complex of finitely generated projective $\mathbb{k}[X]$ -modules whose length l is finite and let ε be an augmentation morphism of P_{\bullet} whose image is R , whence an augmented complex of $\mathbb{k}[X]$ -modules,

$$0 \longrightarrow P_l \longrightarrow P_{l-1} \longrightarrow \cdots \longrightarrow P_0 \xrightarrow{\varepsilon} R \longrightarrow 0$$

Let denote by \mathcal{P}_{\bullet} , \mathcal{R} , \mathcal{K}_0 the localizations on X of P_{\bullet} , R , the kernel of ε respectively and let denote by \mathcal{K}_i the kernel of the morphism $\mathcal{P}_i \longrightarrow \mathcal{P}_{i-1}$ for i positive integer.

Lemma A.1. *Let suppose that S contains the support of the homology of the augmented complex P_{\bullet} .*

- (i) *For all positive integer $i < p - 1$ and for all projective \mathcal{O}_X -module \mathcal{P} , $H^i(X \setminus S, \mathcal{P})$ equals zero.*
- (ii) *For all nonnegative integer $j \leq l$ and for all positive integer $i < p - j$, the cohomology group $H^i(X \setminus S, \mathcal{K}_{l-j})$ equals zero.*

Proof. (i) Let $i < p - 1$ be a positive integer. Since the functor $H^i(X \setminus S, \bullet)$ commutes with the direct sum, it suffices to prove $H^i(X \setminus S, \mathcal{O}_X) = 0$. Since S is a closed subset of X , one has the relative cohomology long exact sequence

$$\cdots \longrightarrow H_S^i(X, \mathcal{O}_X) \longrightarrow H^i(X, \mathcal{O}_X) \longrightarrow H^i(X \setminus S, \mathcal{O}_X) \longrightarrow H_S^{i+1}(X, \mathcal{O}_X) \longrightarrow \cdots$$

Since X is affine, $H^i(X, \mathcal{O}_X)$ equals zero and $H^i(X \setminus S, \mathcal{O}_X)$ is isomorphic to $H_S^{i+1}(X, \mathcal{O}_X)$. Since X is Cohen-Macaulay, the codimension p of S in X equals the depth of its ideal of definition in $\mathbb{k}[X]$ [MA86, Ch. 6, Theorem 17.4]. Hence, according to [Gro67, Theorem 3.8], $H_S^{i+1}(X, \mathcal{O}_X)$ and $H^i(X \setminus S, \mathcal{O}_X)$ equal zero since $i + 1 < p$.

(ii) Let j be a nonnegative integer. Since S contains the support of the homology of the complex P_\bullet , for all nonnegative integer j , one has the short exact sequence of $\mathcal{O}_{X \setminus S}$ -modules

$$0 \longrightarrow \mathcal{K}_{j+1}|_{X \setminus S} \longrightarrow \mathcal{P}_{j+1}|_{X \setminus S} \longrightarrow \mathcal{K}_j|_{X \setminus S} \longrightarrow 0$$

whence the long exact sequence of cohomology

$$\cdots \longrightarrow H^i(X \setminus S, \mathcal{P}_{j+1}) \longrightarrow H^i(X \setminus S, \mathcal{K}_j) \longrightarrow H^{i+1}(X \setminus S, \mathcal{K}_{j+1}) \longrightarrow H^{i+1}(X \setminus S, \mathcal{P}_{j+1}) \longrightarrow \cdots$$

Then, by (i), for $i < p - 2$ positive integer, the cohomology groups $H^i(X \setminus S, \mathcal{K}_j)$ and $H^{i+1}(X \setminus S, \mathcal{K}_{j+1})$ are isomorphic since \mathcal{P}_{j+1} is a projective module. Since $\mathcal{P}_i = 0$ for $i > l$, \mathcal{K}_{l-1} and \mathcal{P}_l have isomorphic restrictions to $X \setminus S$. In particular, by (i), for $i < p - 1$ positive integer, $H^i(X \setminus S, \mathcal{K}_{l-1})$ equal zero. Then, by induction on j , for $i < p - j$ positive integer, $H^i(X \setminus S, \mathcal{K}_{l-j})$ equals zero. \square

Proposition A.2. *Let R' be a $\mathbb{k}[X]$ -module containing R . Let suppose that the following conditions are verified:*

- (1) p is at least $l + 2$,
- (2) X is normal,
- (3) S contains the support of the homology of the augmented complex P_\bullet .

(i) *The complex P_\bullet is a projective resolution of R of length l .*

(ii) *Let suppose that R' is torsion free and let suppose that S contains the support in X of R'/R . Then $R' = R$.*

Proof. (i) Let j be a positive integer. One has to prove that $H^0(X, \mathcal{K}_j)$ is the image of \mathcal{P}_{j+1} . By Condition (3), the short sequence of $\mathcal{O}_{X \setminus S}$ -modules

$$0 \longrightarrow \mathcal{K}_{j+1}|_{X \setminus S} \longrightarrow \mathcal{P}_{j+1}|_{X \setminus S} \longrightarrow \mathcal{K}_j|_{X \setminus S} \longrightarrow 0$$

is exact, whence the cohomology long exact sequence

$$0 \longrightarrow H^0(X \setminus S, \mathcal{K}_{j+1}) \longrightarrow H^0(X \setminus S, \mathcal{P}_{j+1}) \longrightarrow H^0(X \setminus S, \mathcal{K}_j) \longrightarrow H^1(X \setminus S, \mathcal{K}_{j+1}) \longrightarrow \cdots$$

By Lemma A.1, (ii), $H^1(X \setminus S, \mathcal{K}_{j+1})$ equals 0 since $1 < p - l + j + 1$, whence the short exact sequence

$$0 \longrightarrow H^0(X \setminus S, \mathcal{K}_{j+1}) \longrightarrow H^0(X \setminus S, \mathcal{P}_{j+1}) \longrightarrow H^0(X \setminus S, \mathcal{K}_j) \longrightarrow 0$$

Since the codimension of S in X is at least 2 and since X is irreducible and normal, the restriction morphism from \mathcal{P}_{j+1} to $H^0(X \setminus S, \mathcal{P}_{j+1})$ is an isomorphism. Let φ be in $H^0(X, \mathcal{K}_j)$. Then there exists an element ψ of \mathcal{P}_{j+1} whose image ψ' in $H^0(X, \mathcal{K}_j)$ has the same restriction to $X \setminus S$ as φ . Since \mathcal{P}_j is a projective module and since X is irreducible, \mathcal{P}_j is torsion free. Then $\varphi = \psi'$ since $\varphi - \psi'$ is a torsion element of \mathcal{P}_j , whence the assertion.

(ii) Let \mathcal{R}' be the localization of R' on X . Arguing as in (i), since S contains the support of R'/R and since $1 < p - l$, the short sequence

$$0 \longrightarrow H^0(X \setminus S, \mathcal{K}_0) \longrightarrow H^0(X \setminus S, \mathcal{P}_0) \longrightarrow H^0(X \setminus S, \mathcal{R}') \longrightarrow 0$$

is exact. Moreover, the restriction morphism from \mathcal{P}_0 to $H^0(X \setminus S, \mathcal{P}_0)$ is an isomorphism since the codimension of S in X is at least 2 and since X is irreducible and normal. Let φ be in \mathcal{R}' . Then for some ψ in \mathcal{P}_0 , $\varphi - \varepsilon(\psi)$ is a torsion element of \mathcal{R}' . So $\varphi = \varepsilon(\psi)$ since \mathcal{R}' is torsion free, whence the assertion. \square

Corollary A.3. *Let C_\bullet be a homology complex of finite type of $\mathbb{k}[X]$ -modules whose length l is finite and positive. For $j = 0, \dots, l$, let denote by Z_j the space of cycles of degree j of C_\bullet . Let suppose that the following conditions are verified:*

- (1) S contains the support of the homology of the complex C_\bullet ,
- (2) for all i , C_i is a submodule of a free module,
- (3) for $i = 1, \dots, l$, C_i has projective dimension at most d ,
- (4) X is normal and $l + d \leq p - 1$,
- (5) Z_0 equals C_0 .

Then C_\bullet is acyclic and for $j = 0, \dots, l$, Z_j has projective dimension at most $l + d - j - 1$.

Proof. Let prove by induction on $l - j$ that the complex

$$0 \longrightarrow C_l \longrightarrow \dots \longrightarrow C_{j+1} \longrightarrow Z_j \longrightarrow 0$$

is acyclic and that Z_j has projective dimension at most $l + d - j - 1$. For $j = l$, Z_j equals zero since C_l is torsion free by Condition (2) and since Z_l a submodule of C_l , supported by S by Condition (1). Let suppose $j \leq l - 1$ and let suppose the statement true for $j + 1$. By Condition (2), C_{j+1} has a projective resolution P_\bullet whose length is at most d and whose terms are finitely generated. By induction hypothesis, Z_{j+1} has a projective resolution Q_\bullet whose length is at most $l + d - j - 2$ and whose terms are finitely generated, whence an augmented complex R_\bullet of projective modules whose length is $l + d - j - 1$,

$$0 \longrightarrow Q_{l+d-j-2} \oplus P_{l+d-j-1} \longrightarrow \dots \longrightarrow Q_0 \oplus P_1 \longrightarrow P_0 \longrightarrow Z_j \longrightarrow 0.$$

Denoting by d the differentials of Q_\bullet and P_\bullet , the restriction to $Q_i \oplus P_{i+1}$ of the differential of R_\bullet is the map

$$(x, y) \mapsto (dx, dy + (-1)^i \delta(x)),$$

with δ the map which results from the injection of Z_{j+1} into C_{j+1} . Since P_\bullet and Q_\bullet are projective resolutions, the complex R_\bullet is a complex of projective modules having no homology in positive degree. Hence the support of the homology of the augmented complex R_\bullet is contained in S by Condition (1). Then, by Proposition A.2 and Conditions (3) and (4), R_\bullet is a projective resolution of Z_j of length is $l + d - j - 1$ since Z_j is a submodule of a free module by Condition (2), whence the corollary by Condition (5). \square

REFERENCES

- [Au61] M. Auslander, *Modules over unramified regular local rings*, Illinois Journal of Mathematics, **5** (1961), p. 631–647.
- [Bol91] A.V. Bolsinov, *Commutative families of functions related to consistent Poisson brackets*, Acta Applicandae Mathematicae, **24** (1991), n° 1, p. 253–274.
- [Bou02] N. Bourbaki, *Lie groups and Lie algebras. Chapters 4–6. Translated from the 1968 French original by Andrew Pressley*, Springer-Verlag, Berlin (2002).
- [Bou98] N. Bourbaki, *Algèbre commutative, Chapitre 10, Éléments de mathématiques*, Masson (1998), Paris.
- [Bru] W. Bruns and J. Herzog, *Cohen-Macaulay rings*, Cambridge studies in advanced mathematics n° 39, Cambridge University Press, Cambridge (1996).
- [CMo08] J.-Y. Charbonnel and A. Moreau, *Nilpotent bicone and characteristic submodule of a reductive Lie algebra*, Transformation Groups, **14**, (2008).
- [Di74] J. Dixmier, *Algèbres enveloppantes*, Gauthier-Villars (1974).
- [Di79] J. Dixmier, *Champs de vecteurs adjoints sur les groupes et algèbres de Lie semi-simples*, Journal für die reine und angewandte Mathematik, Band. **309** (1979), 183–190.
- [Gi10] V. Ginzburg *Isospectral commuting variety and the Harish-Chandra \mathcal{D} -module* arXiv 1002.20311 [Math.AG].

- [Gi11] V. Ginzburg *Isospectral commuting variety, the Harish-Chandra \mathcal{D} -module, and principal nilpotent pairs* arXiv 1108.5367 [Math.AG].
- [Gro67] A. Grothendieck, *Local cohomology*, Lecture Notes in Mathematics **n°41** (1967), Springer-Verlag, Berlin, Heidelberg, New York.
- [H77] R. Hartshorne, *Algebraic Geometry*, Graduate Texts in Mathematics **n°52** (1977), Springer-Verlag, Berlin Heidelberg New York.
- [HuWi97] G. Huneke and R. Wiegand, *Tensor products of modules, Rigidity and Local cohomology*, *Mathematica Scandinavica*, **81**, (1997), p. 161–183.
- [Ko63] B. Kostant, *Lie group representations on polynomial rings*, *American Journal of Mathematics* **85** (1963), p. 327–404.
- [MA86] H. Matsumura, *Commutative ring theory* Cambridge studies in advanced mathematics **n°8** (1986), Cambridge University Press, Cambridge, London, New York, New Rochelle, Melbourne, Sydney.
- [MF78] A.S. Mishchenko and A.T. Fomenko, *Euler equations on Lie groups*, *Math. USSR-Izv.* **12** (1978), p. 371–389.
- [Mu88] D. Mumford, *The Red Book of Varieties and Schemes*, Lecture Notes in Mathematics **n°1358** (1988), Springer-Verlag, Berlin, Heidelberg, New York, London, Paris, Tokyo.
- [Po08] V.L. Popov *Irregular and singular loci of commuting varieties*, *Transformation Groups* **13** (2008), p. 819–837.
- [Po08] V.L. Popov and E. B. Vinberg, *Invariant Theory*, in: *Algebraic Geometry IV*, *Encyclopaedia of Mathematical Sciences* **n°55** (1994), Springer-Verlag, Berlin, p.123–284.
- [Ri79] R. W. Richardson, *Commuting varieties of semisimple Lie algebras and algebraic groups*, *Compositio Mathematica* **38** (1979), p. 311–322.
- [V72] F.D. Veldkamp, *The center of the universal enveloping algebra of a Lie algebra in characteristic p* , *Annales Scientifiques de L'École Normale Supérieure* **5** (1972), p. 217–240.

UNIVERSITÉ PARIS 7 - CNRS, INSTITUT DE MATHÉMATIQUES DE JUSSIEU, GROUPES, REPRÉSENTATIONS ET GÉOMÉTRIE, CASE 7012, BÂTIMENT CHEVALERET,
75205 PARIS CEDEX 13, FRANCE

E-mail address: jyc@math.jussieu.fr